Intro to Probability

- Series of axioms
  - used to describe uncertainty
- almost everything probabilistically described

Definitions

- Ω → sample space
  1. all (outcomes) in a scenario
     - $\{H, T\}$ or $\{HH, HT, TH, TT\}$

     HH = (ω) $\omega \in \Omega$

     $\omega \in \Omega$

     Possible options 2 coin flips
  2. (Event) → subset of sample space
     - $[A \subseteq \Omega]$
     - $[\{H\}]$ or $[\{HH, TT\}]$
     - (A) → flips where same

- Prob function
  1. $P(\omega)$ = between 0 and 1 → [non neg]
  2. $\left[\sum_{\omega \in \Omega} P(\omega) = 1\right]$ → sum to one

  $0 \leq P(\omega) \leq 1$

- [Uniform] $P(\omega) = \left(\frac{1}{|\Omega|}\right)$

  $|\Omega|$

  - $\{1, 2, 3, 4, \}$
  - 0.5
  - uniform

  $P(\omega) = \frac{1}{|\Omega|} = \frac{1}{4}$

- P(A) of event
  - $\left[\sum_{\omega \in A} (P(\omega))\right]$

  A set

  $P(A) = \frac{|A|}{|\Omega|}$

  uniform sample space

1 Sample Space and Events

Consider the sample space Ω of all outcomes from flipping a coin 3 times.

(a) List all the outcomes in Ω. How many are there?

(b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

(c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

(e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 3 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. [*Hint*: The answer is NOT 1/2.]

$[\Omega]$?

$\omega \to$ coin , side

which

$\omega \; [(HT, H) \; [(HT, T)]] \to (HH, H_1) \; (HH, H_2) \quad [(TT, T_1) \; (TT, T_2)]$

$\Omega \supseteq \Omega_2$

under condition

$[(HT, \cancel{H})] \; [(HH, H_1) \; (HH, H_2)]$

$A \subseteq \Omega_2$

$$P(A) = \frac{|A|}{|\Omega_2|} = \frac{2}{3}$$

$A \to \{HH, H_1 \cdots HH, H_2)$

2 Venn Diagram

$\Omega = \{ \text{Tom}, \text{Bill}, \text{Alex} \cdots$ 1000

$$P(\omega) = \frac{1}{|\Omega|}$$

Out of 1,000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.

$C = \{ \text{Bill}, \text{Alex} \cdots \qquad |C| = 400 \qquad C \cap P = \{ \text{Alex} \cdots$

$P = \{ \text{Tom}, \text{Alex} \qquad |P| = 500 \qquad |C \cap P| = 50$

(b) What is the probability that the student belongs to a club?

$$P(C) = \frac{400}{1000} = 2/5$$

$$\frac{|C|}{|\Omega|}$$

(c) What is the probability that the student works part time?

$$P(P) = 1/2 = \frac{500}{1000}$$

(d) What is the probability that the student belongs to a club AND works part time?

$$P(P \cap C) = \frac{50}{1000}$$

(e) What is the probability that the student belongs to a club OR works part time?

$$P(P \cup C) = \frac{|P \cup C|}{|\Omega|}$$

intersection

$$|P \cup C| = |P| + |C| - |P \cap C|$$

union

$$1/2 \qquad 2/5 \quad - \quad \frac{50}{1000} \quad =$$

3 Counting & Probability

Pedagogy note (Fall 2020): We found students were confused by the assumption of uniformity over solutions here. This is superficially similar to a balls-and-bins problem, but it's more common in balls-and-bins type problems to instead have uniformity over which bin each ball lands in (which leads to a non-uniform distribution over the number of balls in each bin). In general, it's probably a bad idea to include stars-and-bars in probability questions, since it's unusual for "natural" probability problems to take such a form.

Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 70$, where each x_i is a non-negative integer. We choose one of these solutions uniformly at random.

↳ 6 → 5

70 → stars

x_1 0 → +

(a) What is the size of the sample space?

\# stars → 70

\# bars ? 5

$|\Omega| = \binom{75}{5}$

(b) What is the probability that both $x_1 \geq 30$ and $x_2 \geq 30$?

A_2

used stars

↳ how many left?

x_1, x_2

$x_1 = 30$

↳ same bars → 10 stars 70 − 60

$P(A_1 \cap A_2) = \dfrac{|A_1 \cap A_2|}{|\Omega|}$

↳ 15

(c) What it the probability that either $x_1 \geq 30$ or $x_2 \geq 30$?

Ans

$|\Omega| = 2 \cdot 2 \cdot 2 \to 2^3$

a) $\Omega \to \{HHH, (HHT), (TTT) \ldots$ 2^3 $\quad |\Omega| = 2^3 \to 8$

b) start with H $\quad 2^2 \quad A = \{HHH, HHT, HTH, HTT\} \ldots \quad [|A|] = 2^2$

$1 \cdot 2 \cdot 2 \to 2^2$

H $\quad$ H, T

c) End with H $\quad 2^2 \quad B$

d) (start and end) H

$2 \quad [HHH, HTH] \quad C \quad 1 \cdot 2 \cdot 1 \to 2$

e) $|A \cup B| = |A| + |B| - |A \cap B| \to 2^2$ C

$4 + 4 - 2 = 6$

(HHH, HTH) ..... A or B

D $\to$ starts H or ends H

f) A and B not disjoint

$A \cap B \neq \emptyset$

C

$D \to A \cup B$

g) $n \quad |\Omega| = 2^n$

inclusion + exclusion

$|A| = 2^{n-1}$ fix one $\quad |B| = 2^{n-1} \quad |C| = 2^{n-2}$ fix two $\quad |D| = 2 \cdot 2^{n-1} - 2^{n-2}$

$2 \cdot 2 \cdot 2^{n-2}$

$[3 \cdot 2^{n-2}]$

h) options

3 coins

2 sides each

$c_1 \to HH$ (1, 2)

$c_2 \to TT$ (1, 2)

$c_3 \to HT$ (1, 2)

each $\omega \in \Omega$ uniform

(HH, 1) $\to$ we saw heads

(HH, 2)

remove 3 outcomes

(TT, 1) (TT, 2)

(HT, 1 or 2)

wlog

remain [(HH, 1) (HH, 2) (HT, 1)] $\to$ our new space, conditioned on Ω_2

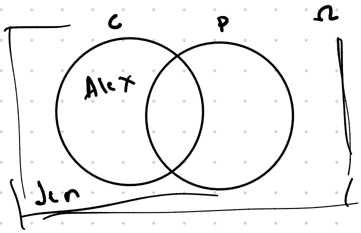
side seen is heads $\to$ other could be heads

A $\to$ heads other side

$\{(HH, 1), (HH, 2)\}$

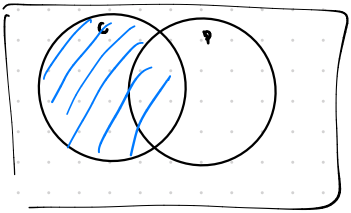
$P(A) = \frac{|A|}{|\Omega_2|} \to \frac{2}{3}$

a)



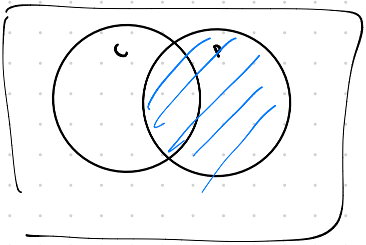
400 → club → C
500 → partying → P
50 → both
1000 → total

b



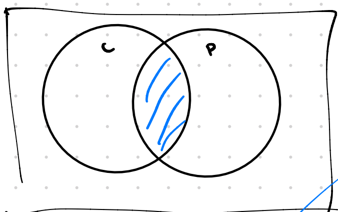
P(C)

↳ $\frac{|C|}{|\Omega|} = \frac{400}{1000} = 2/5$



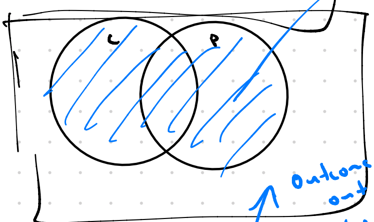
P(P)

↳ $\frac{|P|}{1000} = 1/2$



$P(P \cap C)$

$= \frac{|P \cap C|}{|\Omega|} = \frac{50}{100} = 1/20$



Outcomes out here

$P(P \cup C)$

↳ $P(P) + P(C) - P(P \cap C)$

$1/2 + 2/5 - 1/20 = 17/20$

3a) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 70$

↳ 6 buckets → 5 bars

70 balls → 70 stars

↳ $\binom{75}{5} = |\Omega|$ → $(5 + 30 + 5 + 30 + 0 + 0)$ → $\omega_1 \in \Omega$

3b) $x_1 \geq 30$ $x_2 \geq 30$

↳ place 60 stars 70 − 60 → 10 left

\# ways

$|C|$ → $\binom{10+5}{5}$ $\left[P(C) = \binom{15}{5} / \binom{75}{5} \right]$

3c) $[A \rightarrow x_1 \geq 30]$ → $70 - 30 \rightarrow 40$ $\binom{45}{5}$

$[B \rightarrow x_2 \geq 30]$ ↳ $\binom{45}{5}$

A∩B

↳ $P(A \cup B) = \dfrac{\overset{A}{\binom{45}{5}} + \overset{B}{\binom{45}{5}} - \overset{C}{\binom{15}{5}}}{\binom{75}{5}}$

↳ $|A \cup B|$ ↗