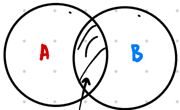
Counting II

- → Stars and Bars
- → Inclusion Exclusion
- → Combinatorial Proofs
  - [case work]

Midterm

- Reflection
- → doubts / q
- notes
  - stuck? more than 5 min move on
  - write literally anything
    - partial credit
  - dont over comp
    - solutions are short
  - → disc problems < exam

Inclusion Exclusion



A and B

disguised into problems be careful

set of possibilities

A

B

A or B = A + B − (A and B)

General

↳ A or B or C ...

$|A| + |B| + |C| \cdots +$ $\binom{n}{1}$

$(-)|A \cap B| - |A \cap C| \cdots$ $\binom{n}{2}$

$(+)|A \cap B \cap C| + \cdots$ $\binom{n}{3}$

$(-)$ $\binom{n}{4}$

$\vdots$

$\binom{n}{n}$

ex

2 coin flips (order matters) → 4 tot ways

ways

at least 1 head: HH [HT TH] $\binom{3}{3}$

at least 1 tail: TT [TH HT]

3 + 3 − |(HT TH)| — # ways ≠ 6

↳ 6 − (2) = 4 ways (all ways)

Stars and Bars

↳ no [replacement]

[order does not matter]

usually apply 2nd rule counting m to 1 → [in no replacement]

ex a dozen bagels where I have 3 types of bagels (onion, bluberry, cheese) — unlimited

↳ dozen (12) = n, choices (3) = k

change view point

Zeroth rule: |way dozen bagel| ⟺ |B| =

↳ 3 bins (for each flavor)

12 balls (throw) ↳ stars

1 poss

4 4 4

onion | blueberry | cheese

or ↳ ||........|.... (8) (4)

12 + 2 ↳ 3 pieces ↳ bar or star

↳ turns into (choose where to put stars or bars)

↳ k = 3 → 2 bars, k − 1 bars

n = 12

↳ bars → $\binom{12+2}{2}$ = or $\binom{12+2}{12}$ ← stars

$\binom{k+n-1}{k}$ or $\binom{k+n-1}{n}$

elements of a sequence are Unique — no replacement

{1, 2, 3, 4, 5} $\binom{5}{3}$

(3!)

(1 2 3)
(2 3 1)
(3 1 2)
(1 3 2)
(2 1 3)
(3 2 1)

6 → {1, 2, 3} order does not matter

3! → 6 to 1 for all

↳ can repeat (replacement) with

(1 1 1) → [{1, 1, 1}] 1 to 1

(1 2 1)
(2 1 1)
(1 1 2) → {1, 1, 2} 3 to 1 — no consist m

↳ diff m to 1 → unordered set

ordered tuple / sequence

Combinatorial Proofs

↳ stories

$\binom{n}{k} = \binom{n}{n-k}$

Obj: Bagel shop n flowers only sell k a day

(1) LHS: the number of ways to choose k things from n ✓

(2) RHS: the shop chooses which flowers not to include

↳ n−k and $\binom{n}{n-k}$ is ways to choose those flowers ✓

Casework

2 coin flips

↳ # of ways

= # of ways conditioned A (first coin H)

+

# of ways conditioned B (first coin T)

st A and B are only cond.

ex

CS 70 Discrete Mathematics and Probability Theory
Summer 2021 **Dis 4A**

1 The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

[2] or [3] or (5) → 6| 10| 15| 30|

2 and 3 2 and 5 3 and 5 ?

↓ ↓ ↓

6 = 2| y| x|

(A ∩ B)

inclusion exclusion →

(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

stars in disguise

(99|7|3|22|0) → stars bars

with replacement
order does not matter

99||7|||...

{0 ... 9}

{9973220} → unordered

⇒ ∃ 1 valid phone number

9973220

⇔ valid phone numbers

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

2 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

cases

what obj of this

(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called Pascal's Identity)

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

3 Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

Answers

1a) Inclusion Exclusion → in first 100 $\mathbb{Z}^+$

↳ A_2 → set div by 2 $\{2, 4 \dots 100\}$

A_3 → $\{3, 6 \dots 99\}$

A_5 → $\{5, 10 \dots 100\}$

$\left\lfloor \frac{100}{6} \right\rfloor$ → floor

2 or 3 or 5 or 4

div 2 and 3 ↳ 6

10 | 15 | 30 |

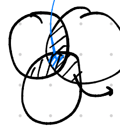
↳ $(|A_2 \cup A_3 \cup A_5|) =$

$|A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5| \dots$

↳ $\left\lfloor \frac{100}{2} \right\rfloor \quad \left\lfloor \frac{99}{3} \right\rfloor \quad \frac{100}{5} - \frac{96}{6} \quad \frac{100}{10} \quad \frac{90}{15} \quad \frac{90}{30}$

3

$(50) + 33 + 20 - 16 - 10 - 6 + 3 = 74$ →



$\binom{n}{2}$

↳ 1 more time time needed

↳ add back

1b) Stars + Bars

$n = 7$ digits

$K = 10$ values $\{0 \dots 9\}$

→ choose digits

[→ only one order]

9 | 8 8 | 7 7 7 | 6 | | | | | | → 0

$\{9, 8, 8, 7, 7, 7, 6\}$ → 9 bars, 7 spots → 10 → $\binom{16}{9}$ → 9 + 7

↓

(only 1 possible tuple)

1c) no repeats

↳ use combinations

↳ choose which 7 to use
only one possible ordering

→ $\{1, 2, 3, 8, 9, 0, 7\}$

↳ (987 3210)

⇒ bijection exist (zeroth rule)

ways to choose 7 digits unique ⟺ number of phone numbers strictly decreasing

2a) Directors from 2n people

↳ obj
assume 2 directors from 2n

$$\left[\binom{2n}{2} = 2\binom{n}{2} + n^2\right]$$

↳ 2n app $\binom{n}{2}$

[LHS: ways to pick 2 directors from 2n applicants]
RHS: split (2n) into 2 groups of (n)

[↳ⁿ CS] [↳ⁿ EECS] → either choose both from (same group) $\binom{n}{2}$ for each group
or diff group $n \cdot n = n^2$
(this covers all poss to choose 2 directors from) 2n
(CS, EECS) (CS, CS) (EECS, EECS)

cases
↳ [split ways into mutually exclusive options that cover all possibilities

① 2 CS $\binom{n}{2}$
② 2 EECS $\binom{n}{2}$
③ 1 CS + 1 EECS
$n \cdot n \longrightarrow n^2$
↳ diff groups ⇒ no replacement

$2\binom{n}{2} + n^2$
sum all cases

2b) Crew members from n
Say we want to choose k

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

LHS choose k members from n people
RHS pick arbitrary member from n people
① accept this person and choose rest k-1 from remaining n-1 for total k
② reject this person; choose k from remaing n-1
↳ cases mutally exclusive and cover all possibilities
⇒ sum and we have chosen k from n

2c) Want to select ~~subset~~ cast from n and 1 from ~~subset~~ cast as lead

$$\sum_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}$$

LHS: assume cast size k from k choose 1 lead (k options)
covers all cast sizes ⇒ counts ways to choose an arbitrary subset + lead

RHS: from n select one lead (n ways) and with remaing construct binary string 1 = in 0 = out each represents a cast

2d) Extend previous to select j leads instead

3) Cases
↳ run starts at ind

0 0 0 0 0

0 or 1

1	1	0	0	0	0	_	_	_	_
0	1	2	3	4	5	6	7	8	9

1 or 0 ↳ 5 coin flips → 4 coin flips; 10 dig string

↑ ↑ ↑ ↑ ↑ ↑ × × × ×

(0): |[5:10]| = 5 → 5
↳ choice 1 or 2 ⇒ 2^5

(1–5): dig before = 1 1x 3 start $2 = 1$ ⇒ 2^4 → 5 − 1

$$\Rightarrow [2^5 + 5 \cdot 2^4] = 112$$