Counting

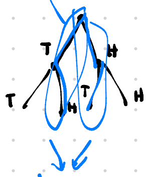
↳ # of possibilities

↳ helpful to know in probability

order no matter [diff → [H H H H ; → T T H ; 5H ; 2T]

T T → diff [2 · 2 · 2 .....]

[(H) T H H ....] ↳ unique n



does matter

(1) With Replacement

Order Matters

↳ [coins → (2^n) ; n flips]

0 1 0 1 ....

n · +

First Rule

↳ k choices

↳ [each choice has a_i options]

coin flips $2 \cdot 2$

# of possibilities

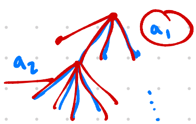
[10 ↓ 1] 5 choices

$\underline{10}$ $\underline{10}$ 10

10 · 9 · 8 · 7 ... 6

↳ [$a_1 \cdot a_2 \cdot a_3 \cdots a_k$]

tree



Second Rule

A → set of options

∃ f m obj in A → 1 obj in B

↳ $|B| = \frac{|A|}{m}$

Without Replacement

Order Matters

↳ (10) cards → # ways to order 5 cards

Permutations $\frac{5 \cdot 4 \cdot 3}{}$

[$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6}$] · (5!

↳ [$\frac{10!}{5!}$] → ($\frac{n!}{(n-k)!}$)

↳

(n) items

(K) to choose

Zeroth Rule

bijection A → B

then |A| = |B|

representations matter!

↳ problem obj has a map to

↳ binary string graph etc

no k to choose

Permute n things

$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdots \underline{2} \cdot \underline{1}$ = (10!)

(n!) k = n

Build (1) without ~~No~~ replacement

↳ Order does $\underline{\text{not}}$ matter

Combinations

($\binom{n}{n}$ ↔ ($\frac{n!}{(n-k)!\,k!}$) = $\binom{n}{k}$

1 2 3

↓

(1)

↳ n things

# of ways to choose a group of k things

n = 3

n! → # 3!

[3 · 2 · 1]

↳ 3!

↳ 3 · 2 · 1

6

10 cards → card hand

[$\frac{1}{5}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{2}$ $\frac{5}{1}$] → (diff permutations]

not 6 3 4 2 1

↳ [1 2 3 ; 2 1 3 ; 3 2 1] (6)

2nd rule of counting ↳ (A) → (B)

(m) to 1

6 to 1

Permutations ↳ ways to order 5 cards → (5!) ⇒ m to 1

combinations B (1)

[$\frac{10!}{5!}$]

$|B| = \frac{(|A|)}{(5!)}$

$|B| = \left(\frac{10}{5! \cdot 5!}\right)$

CS 70 Discrete Mathematics and Probability Theory
Summer 2021 **Dis 3C**

1 Clothing Argument

A B C D → ~~intersection~~

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

$10^4 \rightarrow (10)\cdot(10)\cdot 10 \cdot 10$

(b) How many outfits are there if we wanted to wear exactly two categories?

$a_1 \cdot a_2 \cdot a_3$

$a_2\,(10^2)$

number ways → 2 cat

$[s,t,\ s,h]$ a_1

s,t s,s s,h

$\binom{4}{2} \cdot [10^2]$

→ $10 \cdot 10$

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

10 permute 4

$\left[\binom{10}{4} \cdot 4!\right] \rightarrow$

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

$\binom{10}{4}$

Second rule

$c / 4! \rightarrow d$

m to 1

2 Counting on Graphs + Symmetry

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

$e = [(u,v) = (v,u)]$ undirected

$u, v \in V$

$|V| = n$

0 1 0 1 0 1

$\uparrow \quad \uparrow \quad \uparrow$

$e_1 \quad e_2 \quad e_3$

(b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

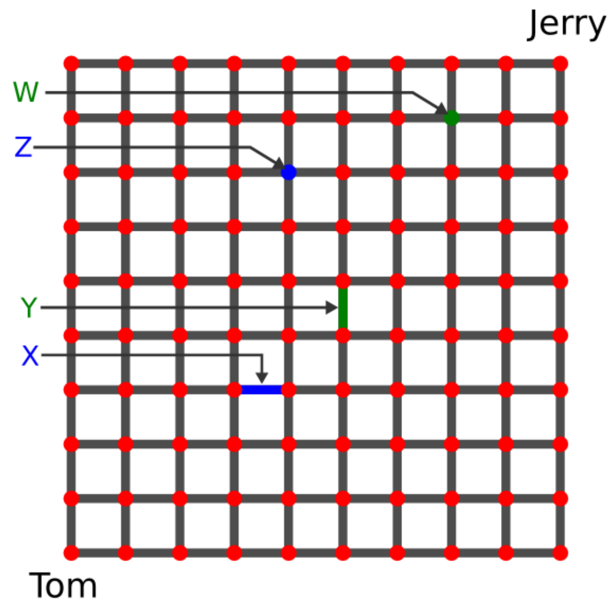
ignore rotations at first

Second rule of counting ?

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

3 Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.



(a) How many such shortest paths exist?

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

Answers

1a) 4 descisions

10 options each

$10 \cdot 10 \cdot 10 \cdot 10 \longrightarrow 10^4$

1b) d ① choose categories → descision 1

$\binom{4}{2}$ options

d ② 10 for 1st cat

d ③ 10 for 2nd cat

$\binom{4}{2} \cdot 10 \cdot 10 \longrightarrow \binom{4}{2} \cdot 10^2$

1c) $\binom{10}{4} \cdot 4!$ → [num ways order n] $n!$ → without replacement

hats, permute

[1st 2nd 3rd 4th (n-1) $4 \cdot 3 \cdot 2 \cdot 1$]

4 3

1d) $\binom{10}{4}$ or $\dfrac{\binom{10}{4} \cdot 4!}{4!}$

hats

permutations

$|V| = n$

$|V| = \{u_1\ u_2\ u_3\}$ (n)

2a) $\left[\binom{n}{2}\right]$ $\left[e = \left[(u,v) = (v,u)\right]\right]$ undirected $\longleftrightarrow \dfrac{n!}{(n-2)! \cdot \cancel{2!}\ 2}$ → $\left(\dfrac{n(n-1)}{2}\right)$

n

choose the incident vertex pair out of n

2b) K → num vertex in cycle → n vertex ; fully connected [any group chosen has a cycle]

no repeats

min 3

max n

$\binom{n}{k} \cdot k!$

divide out dupl

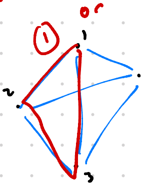
→ rotations (k)

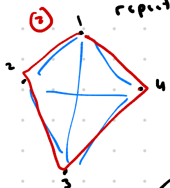
123 231 312

→ inversions (2)

123 321

① or ② ...





$\left(\begin{matrix} 1\ 2\ 3 \\ 2\ 3\ 1 \\ 3\ 2\ 1 \\ \vdots \end{matrix}\right)$ $\left(\begin{matrix} 1\ 2\ 3\ 4 \\ 4\ 3\ 2\ 1 \\ 3\ 2\ 1\ 4 \end{matrix}\right)$ → [sequences same]

→ (not 4 2 1 3) a diff cycle

$\hookrightarrow \binom{n}{k} \cdot k! \cdot \frac{1}{2k} \quad \Sigma \rightarrow$ sum for all k $\sum_{k=3}^{n} \left[\binom{n}{k} \cdot k!\right] \cdot \frac{1}{2k}$

$\hookrightarrow \frac{n!}{(n-k)!}$ (permutations)

2c) no sym (dup) (rotations)

$\hookrightarrow$ n! order n things

$\hookrightarrow$ n! / n $\rightarrow$ (n-1)!

$\hookrightarrow$ [n rotations for [one bracelet]]

n! $\rightarrow$ permutations

A n to 1 B

$\hookrightarrow$ (n) permutations to brackets

2d) 6! ways (no sym) dup

$\hookrightarrow$ 6! / [rotations per coloring] $\rightarrow$ 6! / 6·4

(6!)

g y b
1 2 3 4 5 6
top bot r l

$\hookrightarrow$ [face can be in 6 places] · [at each place cube rotate into 4 ways on one axis]

3a) 4 ↑ 4 →

shortest path in 18 spots (4+4)

$\hookrightarrow$ any ordering of

↑ ↑ ↑ → → → ....

choose ↑ 4

$\hookrightarrow \binom{18}{4}$ ↑ _ ↑

3b) get to x vertex

$\hookrightarrow$ 3↑ 3→

$\hookrightarrow \binom{6}{3}$

$\hookrightarrow$ get to Jerry

6↑ 5→ $\rightarrow \binom{11}{5}$ or $\binom{11}{6}$

$\hookrightarrow$ x is → $\Rightarrow \binom{6}{3} \cdot \binom{11}{5}$

neither

$\hookrightarrow$ total − (through x) or (through y)

$\hookrightarrow \binom{18}{9} - x + y - x$ and y

y

$\hookrightarrow$ 4↑ 5→ $\Rightarrow \binom{9}{5} \cdot \binom{8}{4}$

4↑ 4→

both

$\hookrightarrow$ to x $\binom{6}{3}$ x to y 1↑ 1→ $\binom{2}{1}$ y to Jerry $\binom{8}{4}$

$\binom{6}{3} \cdot \binom{2}{1} \cdot \binom{8}{4}$