

Sets, Functions, Countability

**Sets**

a group of distinct elements

order repetition does not matter

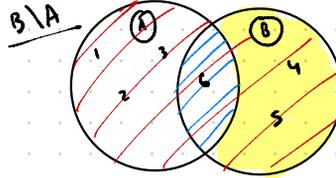
anything colors etc

$A = \{1, 2, 3, 33\}$   
 $B = \{1, 2, 33\}$

$A = B$   
 Same set

$A = \{1, 2, 3\}$   
 $U$   
 $B = \{1, 2, 33\}$

$\{1, 2, 3, 33\}$   
 $\{1, 2, 3\}$



$(A \cap B)$   
 $\forall x \in (A \cap B)$   
 $x \in A \wedge x \in B$

**Operations Set**

$\cap$  intersection

$\cup$  union

$\setminus$  or  $-$  relative complement

$A = \{1, 2, 3\}$   $B = \{2, 4, 3\}$  or [set diff]

Sets of Collection  $\{ \{1, 3\}, \{2, 4\} \}$

Cartesian Product  
 $A \times B = \{ (a, b) \mid a \in A, b \in B \}$   
 ordered tuples

all pairs set  
 A term then B term  
 in pair

Power set of A  
 all possible subsets of A

any  $S \subseteq A$   
 ex  $A = \{1, 2, 33\}$   
 $P(A) = \{ \emptyset, \{1, 2, 33\} \}$

$\emptyset, \{1, 3\}, \{2, 3\}, \{1, 2, 33\}$

$\cap$  intersection

$\cup$  union

$(B \setminus A)$  or  $B - A$   
 (relative comp) (set diff)

$x \in A \wedge x \in B$   
 $x \in A \vee x \in B$   
 $[x \in B \mid x \notin A]$  for such that

$B \cap A$   
 $B \cup A$   
 cont

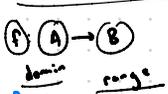
Subsets

$A \subseteq B$  (if B contains all values in A) in B  
 Proper Subset  
 Larger set TDR  
 $A = B$   
 $A$  and  $B$   
 $\{1, 2, 3\}$

$A \subset B$  (if B contains all values in A) and  
 B has at least 1 element A does not  
 $A \neq B$   
 $A \{1, 2, 3\}$   
 $B \{1, 2, 3, 33\}$   
 $A \subset B$

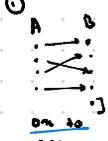
**Functions**

mapping two sets

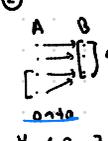


$f: Z \rightarrow Z$   
 all  $\forall a \in A$  there exist exactly one  $b \in B$  mapped to a

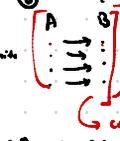
- ① injective  $\rightarrow$  one to one
- ② surjective  $\rightarrow$  onto
- ③ bijective  $\rightarrow$  both



$\forall (x, y) \in A$   
 $x \neq y \Rightarrow f(x) \neq f(y)$

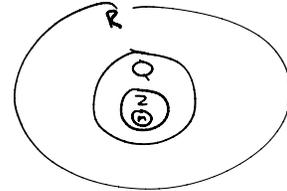


$\forall y \in B \exists x \in A$  st  $f(x) = y$



bijective  
 Both onto and one to one  
 $\Rightarrow |A| = |B|$

1 Set Operations



- $\mathbb{R}$ , the set of real numbers
- $\mathbb{Q}$ , the set of rational numbers:  $\{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$
- $\mathbb{Z}$ , the set of integers:  $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{N}$ , the set of natural numbers:  $\{0, 1, 2, 3, \dots\}$

(a) Given a set  $A = \{1, 2, 3, 4\}$ , what is  $\mathcal{P}(A)$  (Power Set)?

$$\mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \}$$

↳ values      ↳ (subsets of A)

(b) Given a generic set  $B$ , how do you describe  $\mathcal{P}(B)$  using set comprehension notation? (Set Comprehension is  $\{x \mid x \in A\}$ .)

$$\mathcal{P}(B) = \{ S \mid S \subseteq B \}$$

↳ any subset of B is an element in power set

(c) What is  $\mathbb{R} \cap \mathcal{P}(A)$ ?

↳  $\emptyset, \{2\}$  (empties)  
 ↳ what is a power set?  $\mathbb{R} = \{1, 1.2, 1.7, \dots\}$   
 ↳ no sets in  $\mathbb{R}$

(d) What is  $\mathbb{R} \cap \mathbb{Z}$ ?

$$\mathbb{Z} \subseteq \mathbb{R} \Rightarrow \mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$$

(e) What is  $\mathbb{N} \cap \mathbb{Q}$ ?

$$\mathbb{N} \subseteq \mathbb{Q} \Rightarrow \mathbb{N} \cap \mathbb{Q} = \mathbb{N}$$

↳ can unique

(f) What kind of numbers are in  $\mathbb{R} \setminus \mathbb{Q}$ ?

↳ the set of irrational numbers  
 ↳ are real but not rational

(g) If  $S \subseteq T$ , what is  $S \setminus T$ ?

↳  $S \subseteq T \Rightarrow S \setminus T = \emptyset$   
 ↳ all values of S in T  
 ↳ cannot exist  $x \in S$  but  $x \notin T$

Countability if  $\exists$  a bijection  $\mathbb{N}$  and set  $S$  the  $S$  is countable

↳ All finite sets are countable  
 ↳ all uncountable sets are infinite

Countably finite  
 $\mathbb{N} = \{0, 1, 2, \dots\}$   
 $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$   
 $\mathbb{Z}^+ \subseteq \mathbb{N}$

10 min

## 2 Bijections

Consider the function

onto 0.25 0.3

$$f(x) = \begin{cases} [x, ] & \text{if } x \geq 1; \text{ } \in \mathbb{Q} \\ \frac{x^2}{2}, ] 0 & \text{if } -1 \leq x < 1; \\ \lfloor 2x+3 \rfloor & \text{if } x < -1. \end{cases}$$

identity fun -  $\rightarrow$  R = [-∞, ∞]

↳ skip and 2-1 + 3 = 1

(a) If the domain and range of  $f$  are  $\mathbb{N}$ , is  $f$  injective (one-to-one), surjective (onto), bijective?

Bijective

(b) If the domain and range of  $f$  are  $\mathbb{Z}$ , is  $f$  injective (one-to-one), surjective (onto), bijective?

not injective  $x = -1 \ y = 1$  not one to not surjective

$f(x) = f(y) \rightarrow$  bijective

↳ skip every even

(c) If the domain and range of  $f$  are  $\mathbb{R}$  is  $f$  injective (one-to-one), surjective (onto), bijective?

↳ surjectivity holds  $\rightarrow$   $f$  continuous if  $\uparrow$  injective  $\Rightarrow \uparrow$  bijective

injection  $\rightarrow$

↳ check  $\epsilon$  for surjectivity

## 3 Unions and Intersections

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a)  $A \cap B$ , where  $A$  is countable, and  $B$  is uncountable

Always Countable  $A \cap B \subseteq A \Rightarrow$  since  $A$  countable  $A \cap B$  countable

(b)  $A \cup B$ , where  $A$  is countable, and  $B$  is uncountable

Always uncountable  $B \subseteq A \cup B \Rightarrow$  since  $B$  uncountable  $A \cup B$  uncountable

(c)  $\bigcap_{i \in A} S_i$  where  $A$  is a countable set of indices and each  $S_i$  is an uncountable set.

Sometimes both based on  $S_i$

↳ Countable  $\Rightarrow S_i$  disjoint

Un-countable  $\Rightarrow S_i$  are identical

# Answers



2a) Identity  
injection, surjection, bijective

2b) injective → not surjective  
↳ linear space two ones  $y=1$   
↳ not bijective

2c)

3a) always count  $A \cap B \subseteq A$

$$|A \cap B| \leq |A|$$

3b) always uncount  $B \subseteq A \cup B$

$$|A \cup B| \geq |B|$$

3c) Sometimes