Joint Distributions

The relationship between the occurrences of two random variables

X, Y A B $P(A \cap B)$

pmf ex ~ pdf

$[P(X = i), (Y = j)]$ → pmf → [prob mass function] joint dist of X, Y discrete

$f_{x,y}(x, y)$ joint dist pdf of X, Y continuous

$P(\underbrace{a \le X \le b}_{A}, \underbrace{c \le Y \le d}_{B})$ → $\int_c^d \int_a^b f_{xy}(x,y)\,dx\,dy$

① Important Properties

as any other dist joint dist sums to 1

[non zero prob] $[X \in [0,1]]$ $[Y \in [0,1]]$

$$\int_0^1 \int_0^1 f_{xy}(x,y)\,dx\,dy = 1$$

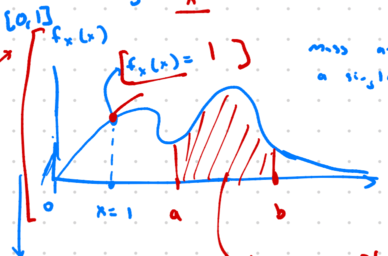
cxy → $\frac{c}{4} = 1$, $c = 4$

$f_{xy}(x,y) = cxy$, $X, Y \in [0,1]$

$f_{xy}(x,y) \to$ pdf, $-\infty$ ∞

$$\left[\sum_{j \in Y} \sum_{i \in X} P(X = i, Y = j)\right] = 1$$

visually



mass at a single point

area $P(X \in [a,b])$

$f_{x,y}(x,y)$

marginals

$$f_x(x) = \int_0^1 f_{xy}(x,y)\,dy$$

$Y \in [0,1]$ bounds of y st non zero prob

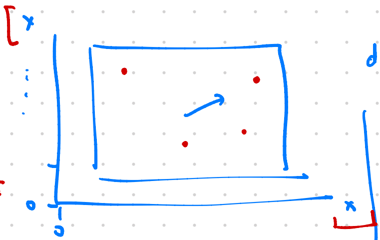
1x $2xy$ $[Y \in [0,1], X \in [0,1]]$

$$f_x(x) = \int_0^1 (2xy)\,dy = 2x \cdot \frac{1}{2}$$

$f_x(x) = x$

$f_y(y) = y$

$2xy \ne xy$



depth out of page

if $f_{x,y}(x,y)$ prob mass

joint is a prob mass plane cdf → volume

conditional pdfs

bayes rule

$$\left[f_{x \mid Y = y}(x)\right] = \frac{f_{xy}(x,y)}{f_y(y)}$$

$[P(X \in [a,b] \mid Y = y)]$

$$= \left[\int_a^b f_{x \mid Y = y}(x)\right]$$

highest

$f_{x \mid Y = y}(x)$ $\left[\frac{f_{xy}(x,y)}{f_y(y)}\right]$

$f(x) \to$

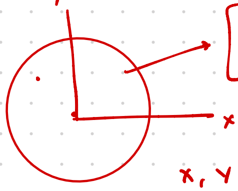
① ind

$$\underbrace{f_{xy}(x,y)}_{\text{joint pdf}} = \underbrace{f_x(x)\, f_y(y)}_{\text{marginal pdfs}}$$

↳ Visual examples

Disk

↳



[$f_{X,Y}(x,y)$ constant in circle] ↳ 0 outside

$x, y \to$

↳ $[x^2 + y^2 = r^2]$ outer points

↳ area is any piece within

$\to [x^2 + y^2 \le r^2]$

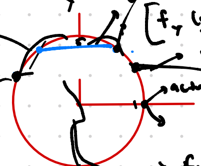
$\iint f_{xy}(x,y) = 1$ → c

$[\forall (x,y) \mid x^2+y^2 \le r^2] \to c \cdot \left[\iint_{x^2+y^2 \le r^2} 1 \, dx\,dy\right] = 1$ → area of circle

$(c \pi r^2) = 1$

$f_{xy}(x,y) = \dfrac{1}{\pi r^2}$

① Marginals



bound when cond

$[f_Y(y)]$

y = 2

actual

$\pm\sqrt{r^2-y^2}$

mass under curve

$f_Y(y)$

weight of this line

~ $P(Y=y)$

↳ integrate with x

[→ bounds of x

$[(x^2 + (y^2)) = r^2$

$x = \pm\sqrt{r^2-y^2}$]]

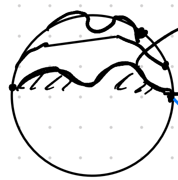
given y

bounds of x ←

$f_X(x) = \int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} f_{xy}(x,y)\,dx$

→

② conditional



$[f_{X \mid Y=y}(x)] \to$

↳ $\left[\dfrac{f_{XY}(x,y)}{f_Y(y)}\right] \to \left[\dfrac{1}{2\sqrt{r^2-y^2}}\right]$

[constant intervals of diff len]

depends on y ⇒

↳ $[f_{xy}(x,y) = f_x(x) \cdot f_y(y)$

$f_{X \mid Y=y}(x) \neq f_X(x)$

and not ind]

Mixed Joint Distributions

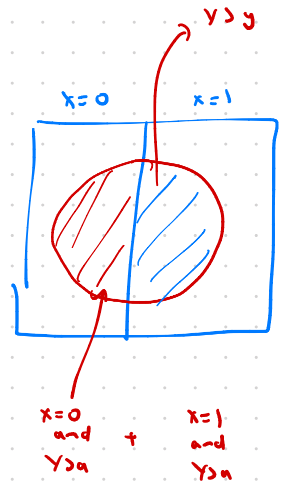
↳ discrete and cont

to find prob use total prob theorem

↳ $X \sim \text{Bern}(p)$

$$Y \sim \begin{cases} \exp(\lambda_1) & \text{if } X=1 \\ \exp(\lambda_2) & X=0 \end{cases}$$

joint total law



pdf of Y?

↳ cdf → event $Y > y$

$$P(Y>y) = P(Y>y \cap X=1) + P(Y>y \cap X=0)$$

↳ $\frac{d}{dy} P(Y>y) \to f_Y(y)$ pdf

Overall Approach

↳ draw graph of joint dist

↳ might make finding certain prob easier

Q1 Sp 19 Final

1) Consider a cont rand var X st

$f_X(x) = c x^{-3}$ for $x \geq 1$

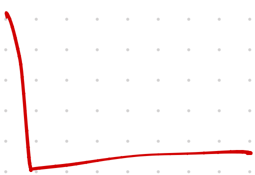
$\hookrightarrow 0$ otherwise

$\hookrightarrow$ what is c?

2) Consider X, Y $f_{XY}(x,y) = cxy$ $x,y \in [0,1]$

$\hookrightarrow 0$ outside

$\hookrightarrow$ what is c?



Answ

1) $\int_1^\infty c x^{-3} = 1 \longrightarrow -\frac{c x^{-2}}{2} \Big|_1^\infty = \frac{c}{2} = 1$

2a) $\int_0^1 \int_0^1 cxy \, dx \, dy = \frac{c}{4} = 1$ $c = 4$ $c = 2$

2b)

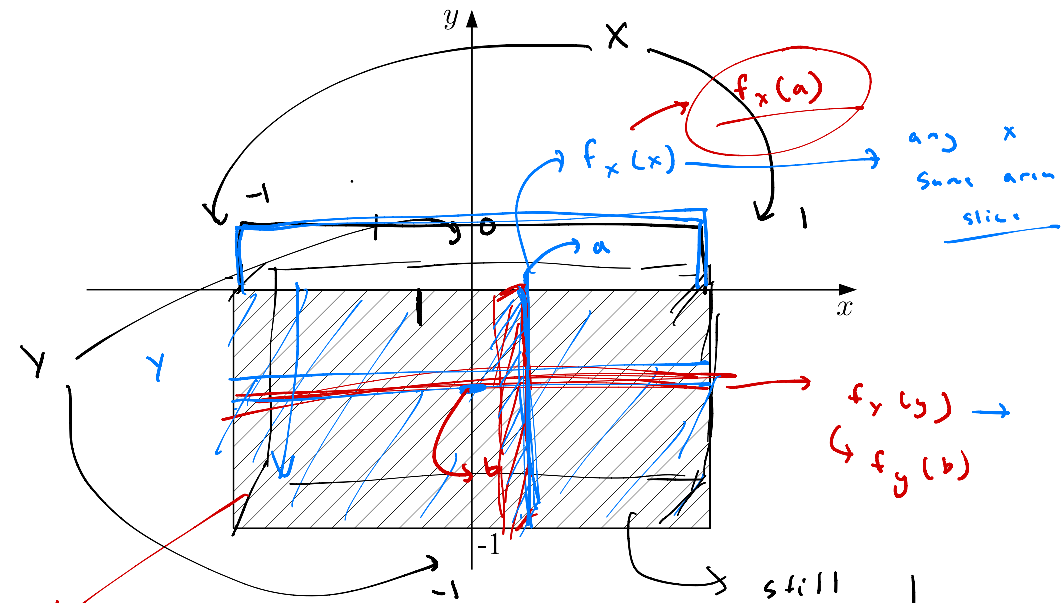
3 (6 Points) Joints

Let X and Y be two continuous random variables. The joint density of (X,Y) is uniform on the shaded region below, and 0 outside the shaded region. Mathematically, the figure consists of a rectangle.

(a) (1 point) What is the joint density $f_{X,Y}$ on the shaded region? → assumption → 1

(b) (2 points) Set up, but do not evaluate the integrals for the values of $f_X(x)$ and $f_Y(y)$ on the shaded region.

(c) (3 points) Are X and Y independent? **Justify your answer.**



find height

1 unit prob mass

Answ

1) $\begin{bmatrix} f_{xy}(x,y) \to \text{uniform} \\ \text{volume} \to 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \text{for an area} \\ \text{of 2 units} \end{bmatrix}$ $\quad 2 \cdot f_{xy}(x,y) = 1$

$\hookrightarrow \frac{1}{2}$

2) $f_x(x) = \int_{-1}^{0} \overset{1/2}{f_{xy}(x,y)}\, dy \quad \frac{1}{2} y \Big|_{-1}^{0} \to \frac{1}{2}$ $\quad y$

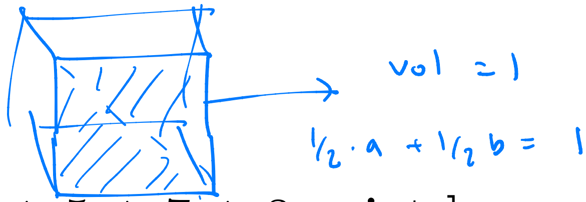
$f_y(y) = \int_{-1}^{1} \overset{1/2}{f_{xy}(x,y)}\, dx \to 1$

3) Yes as neither density depends on the value of the other variable

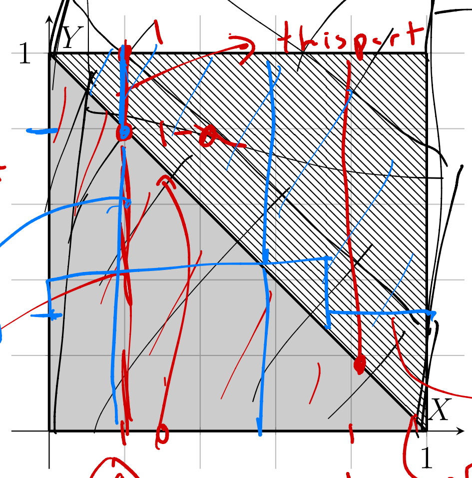
$\hookrightarrow f_x(x) \cdot f_y(y) = f_{xy}(x,y) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$f_{x|Y=y}(x) = f_x(x)$ ind

$\hookrightarrow \left[\dfrac{f_{xy}(x,y)}{[f_Y(y)]} \right]$



3 Graphical Density [3 + 3 + 5 + 7 + 8 points]



$\iint = 1$

$f_X(x)$ a b

$f_{XY}(x,y) = \begin{cases} 3A \\ A \end{cases}$ → $f_{XY}(x,y)$

X, Y ~ cont uniform

$3A \cdot \frac{1}{2} + A \cdot \frac{1}{2} = 1$

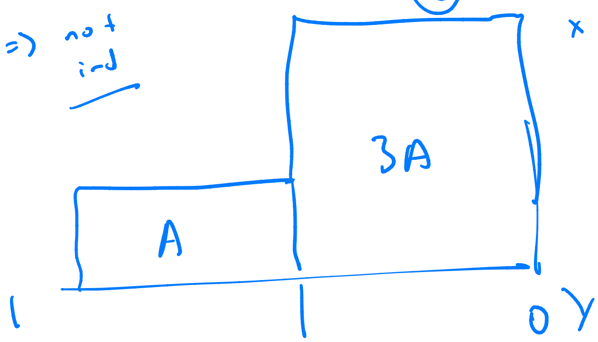
$A = \frac{1}{2}$

(a) Are X and Y independent? Remember to justify your answer.

$y = 1 - x$

$x = 1 - y$

$f_{X|Y=y}(x)$ based y $\neq f_X(x)$ ⇒ not ind

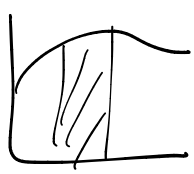


(b) What is the value of A?

$A = \frac{1}{2}$

1 − a

(c) Compute $f_X(x)$.



3A $\cdot \frac{1}{2}$

$f_{XY}(x,y) = \begin{cases} 3A \to 3/2 \\ A \to 1/2 \end{cases}$

$\iint 1 \to$ area

$\iint f_{XY}(x,y) = 1$

c

$\frac{1}{2} 3A + \frac{1}{2} A = 1 \quad A = \frac{1}{2}$

Answ

a) dist of Y must be same regardless of X for ind

we can see however that the $E[Y \mid X=i]$ is not constant for all i

$X=0$ $E[Y] = 1/2$ → constant density

$X = 1/2$ → changes less than since more dense in bottom

gray triang

dashed triangl

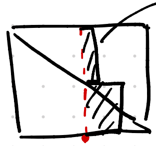
$$\iint f_{XY}(x,y) + \iint f_{XY}$$

b) total area → 1

$$\tfrac{1}{2}(3A) + \tfrac{1}{2}A = 1 \qquad A = 1/2$$

density

c) $f_X(x)$ →



$$\int_0^1 f_{XY}(x,y)\,dy$$

$$\int_0^{1-x} 3 \cdot A \, dy + \int_{1-x}^{1} 1 \cdot A \, dy$$

1/2

top A

bottom grey → 3A

$$\tfrac{3}{2}(1-x) + \tfrac{1}{2}(x)$$

$$= 3/2 - x$$