Joint Distributions

The relationship between the occurrences of two random variables

X, Y A   B $P(A \cap B)$ → [prob mass function] pmf

pmf ex ~ pdf

$[P(X = i, Y = j)]$ joint dist of X, Y discrete

$f_{X,Y}(x,y)$ joint dist pdf of X, Y continuous

$P(a \le X \le b,\ c \le Y \le d)$ → $\int_c^d \int_a^b f_{XY}(x,y)\,dx\,dy$

visually

① Important Properties

as any other dist, joint dist sums to 1

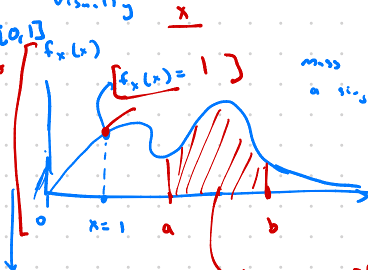
[non zero prob] $[X \in [0,1]]$ $[Y \in [0,1]]$

$$\int_0^1 \int_0^1 f_{XY}(x,y)\,dx\,dy = 1$$

$f_{XY}(x,y) \to$ pdf $-\infty$ ∞

$$\left[\sum_{j \in Y} \sum_{i \in X} P(X=i, Y=j)\right] = 1$$

$f_{XY}(x,y) = cxy$ $X, Y \in [0,1]$ cxy → $\frac{c}{4} = 1$ $c = 4$



mass at a single point

X

area $P(X \in [a,b])$

marginals

$$f_X(x) = \int_0^1 f_{XY}(x,y)\,dy$$

$Y \in [0,1]$ → bounds of y st non zero prob

1x   2xy $\begin{bmatrix} Y \in [0,1] \\ X \in [0,1] \end{bmatrix}$

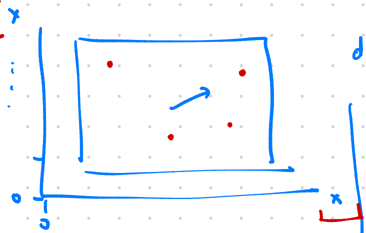
$$f_X(x) = \int_0^1 (2xy)\,dy = 2x \cdot \frac{1}{2}$$

$f_{XY}(x,y)$

$f_X(x) = x$

$f_Y(y) = y$

$2xy \neq xy$



depth out of page

if $f_{X,Y}(x,y)$ → prob mass

→ joint is a prob mass plane   cdf → volume

conditional pdfs

bayes rule

$$\left[f_{X \mid Y=y}(x)\right] = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$[P(X \in [a,b] \mid Y = y)] = \left[\int_a^b f_{X \mid Y=y}(x)\right]$$

highest

$\int$

$f_{X \mid Y=y}(x)$ $\left[\frac{f_{XY}(x,y)}{f_Y(y)}\right]$

$f(x) \to$

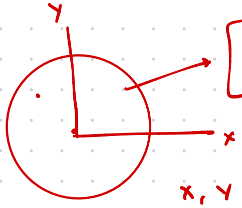
① ind

$$\underbrace{f_{XY}(x,y)}_{\text{joint pdf}} = \underbrace{f_X(x)\ f_Y(y)}_{\text{marginal pdfs}}$$

↳ Visual examples

Disk

↳



[$f_{X,Y}(x,y)$ constant in circle]

↳ 0 outside

x, y →

↳ [$x^2 + y^2 = r^2$] outer points

↳ area is any piece within

[$x^2 + y^2 \le r^2$]

c

$$\iint f_{xy}(x,y) = 1$$

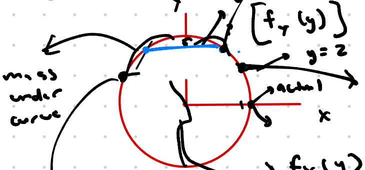
[$\forall (x,y) \mid x^2+y^2 \le r^2$] → $c \cdot \left[\iint_{x^2+y^2 \le r^2} 1 \, dx\,dy \right] = 1$

area of circle

$(c \pi r^2) = 1$

$$f_{xy}(x,y) = \frac{1}{\pi r^2}$$

(1) Marginals



bound when cond

[$f_Y(y)$]

$\pm\sqrt{r^2 - y^2}$

$f_Y(y)$

weight of this line

~ $P(Y=y)$

↳ integrate with x →

[→ bounds of x

[$(x^2 + (y^2)) = r^2$

$x = \pm\sqrt{r^2 - y^2}$]]

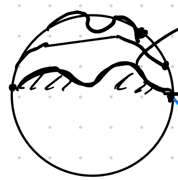
given y

bounds of x &

$$f_x(x) = \int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} f_{xy}(x,y)\,dx$$

(2) conditional

[$f_{X|Y=y}(x)$] →



the function

y =

constant intervals of diff len

depends on y

⇒

↳ [$\frac{f_{XY}(x,y)}{f_Y(y)}$] → [$\frac{1}{2\sqrt{r^2 - y^2}}$]

↳ $f_{xy}(x,y) = f_x(x) \cdot f_y(y)$

$f_{X|Y=y}(x) \neq f_X(x)$

and not ind

Mixed Joint Distributions

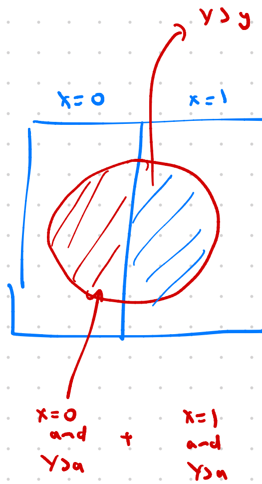
↳ discrete and cont

to find prob use total prob theorem

↳ $X \sim Bern(p)$

$$Y \sim \begin{cases} \exp(\lambda_1) & \text{if } X=1 \\ \exp(\lambda_2) & X=0 \end{cases}$$

joint total law



pdf of Y?

↳ cdf → event $Y > y$

$$P(Y>y) = P(Y>y \cap X=1) + P(Y>y \cap X=0)$$

↳ $\frac{d}{dy} P(Y>y) \rightarrow f_Y(y)$ pdf

Overall Approach

↳ draw graph of joint dist

↳ might make finding certain prob easier

Q1 Sp 19 Final

1) Consider a cont rand var X st

$f_X(x) = c x^{-3}$ for $x \geq 1$

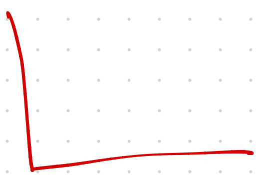
0 otherwise

what is c?

2) Consider X, Y $f_{XY}(x,y) = cxy$ $x,y \in [0,1]$

0 outside

what is c?



Answ

1) $\int_1^\infty c x^{-3} = 1 \rightarrow -\frac{c x^{-2}}{2} \Big|_1^\infty = \frac{c}{2} = 1$

2a) $\int_0^1 \int_0^1 cxy \, dx \, dy = \frac{c}{4} = 1$ $c = 4$ $c = 2$

2b)

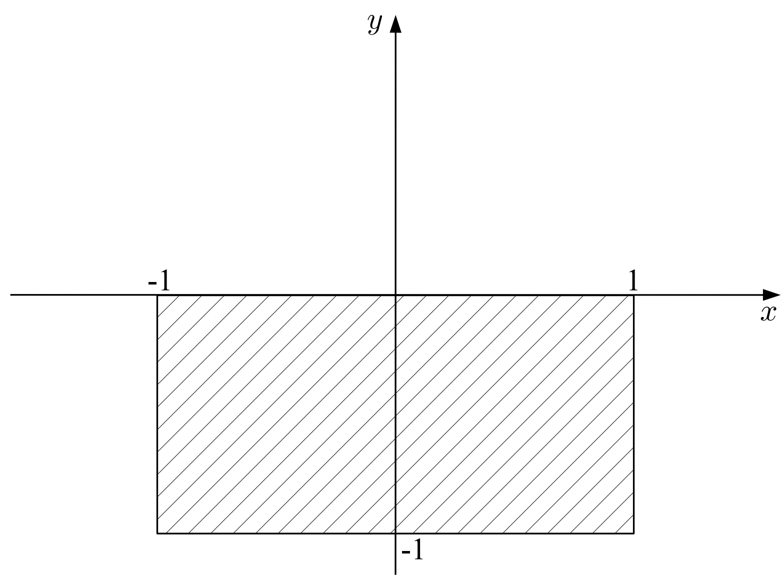
3 (6 Points) Joints

Let X and Y be two continuous random variables. The joint density of (X,Y) is uniform on the shaded region below, and 0 outside the shaded region. Mathematically, the figure consists of a rectangle.

(a) (1 point) What is the joint density $f_{X,Y}$ on the shaded region?

(b) (2 points) Set up, but do not evaluate the integrals for the values of $f_X(x)$ and $f_Y(y)$ on the shaded region.

(c) (3 points) Are X and Y independent? **Justify your answer.**



Answ

1) $\left[\begin{array}{l} f_{XY}(x,y) \to \text{uniform} \\ \text{volume} \to 1 \end{array}\right] \Rightarrow \left[\begin{array}{l} \text{for an area} \\ \text{of 2 units} \end{array}\right]$ $2 \cdot f_{XY}(x,y) = 1$ $\hookrightarrow \frac{1}{2}$

2) $f_X(x) = \int_{-1}^{0} f_{XY}(x,y)\, dy \quad \frac{1}{2} y \Big|_{-1}^{0} \to \frac{1}{2}$ $\quad y$

$f_{XY}(x,y) \to \frac{1}{2}$

$f_Y(y) = \int_{-1}^{1} f_{XY}(x,y)\, dx \to 1$

$f_{XY}(x,y) \to \frac{1}{2}$

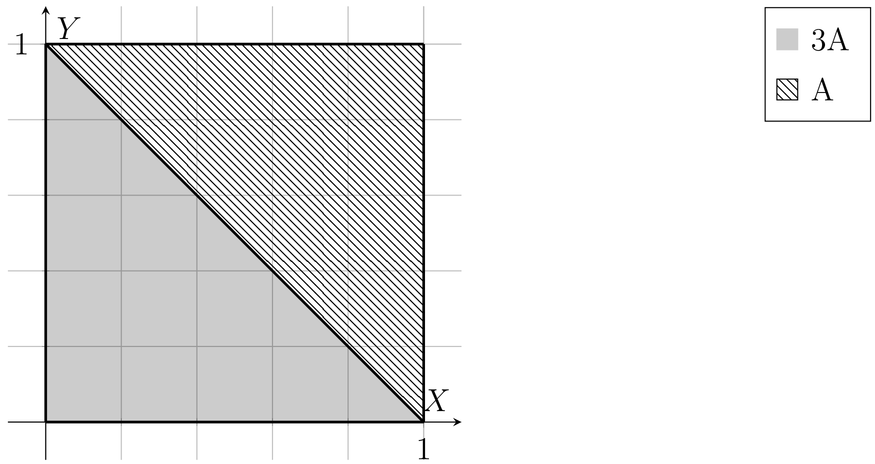
3) Yes as neither density depends on the value of the other variable

$\hookrightarrow f_X(x) \cdot f_Y(y) = f_{XY}(x,y) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$f_{X|Y=y}(x) = f_X(x)$ ind

$\hookrightarrow \left[\dfrac{f_{XY}(x,y)}{[f_Y(y)]} \right]$

3 Graphical Density [3 + 3 + 5 + 7 + 8 points]



(a) Are X and Y independent? Remember to justify your answer.

(b) What is the value of A?

(c) Compute $f_X(x)$.

Answ

a) dist of Y must be same regardless of X $\underbrace{\text{for ind}}$

we can see however that the $E[Y \mid X = i]$ is not constant for all i

$X = 0$ $E[Y] = 1/2$ → constant density

$X = 1/2$ → changes less than since more dense in bottom

gray triang dashed triangl

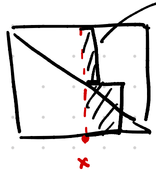
$$\iint f_{XY}(x,y) + \iint f_{XY}$$

b) total area → 1

$$\frac{1}{2}(3A) + \frac{1}{2}A = 1 \qquad A = 1/2$$

density

c) $f_X(x)$ →



↳ $f_X(x)$

$$\int_0^1 f_{XY}(x,y)\,dy$$

$$\int_0^{1-x} 3 \cdot A \, dy + \int_{1-x}^{1} 1 \cdot A \, dy$$

bottom grey ↳ 3A

1/2

top A

$$\frac{3}{2}(1-x) + \frac{1}{2}(x)$$

$$= 3/2 - x$$